11 Homomorfizmi grup.

1. Razloži, zakaj preslikava

$$\phi : \mathbb{Z}/\langle 3\rangle \longrightarrow \mathbb{Z}_6$$
$$x + \langle 3\rangle \longrightarrow 3x$$

ni dobro definirana.

2. Pokaži, da preslikava $\phi : \mathbb{R} \to \mathbb{R}$, iz grupe realnih števil glede na operacijo seštevanja nase, definirana z $\phi(x) = x^2$, ni homomorfizem.

Definicija (homomorfizem)

Homomorfizem ϕ iz grupe G v grupo $\overline{G}$ je preslikava iz G v $\overline{G}$, ki ohranja binarno operacijo grupe; torej $\phi(ab) = \phi(a)\phi(b)$ za vse $a, b \in G$.

Definicija (jedro homomorfizma)

Jedro homomorfizma ϕ iz grupe G v grupo z ideniteto e je množica $\{x \in G \,|\, \phi(x) = e\}$. Jedro homomorfizma ϕ označujemo s kerϕ.

3. Naj bo $\mathbb{R}^*$ grupa neničelnih realnih števil glede na operacijo množenja.

(i) Pokaži, da je preslikava $\phi(A) = \det(A)$ homomorfizem iz $\mathrm{GL}_2(\mathbb{R})$ v $\mathbb{R}^*$. Določi jedro homomorfizma ϕ.

(ii) Pokaži, da je preslikava ϕ iz $\mathbb{R}^*$ v $\mathbb{R}^*$, ki je definirana s $\phi(x) = |x|$, homomorfizam. Določi jedro homomorfizma ϕ.

(iii) Ali je preslikava $\phi(x) = x^2$ iz $\mathbb{R}^*$ nase, homomorfizem? Če je, določi jedro homomorfizma ϕ.

4. Naj bosta G in H dani grupi.

(i) Pokaži, da je preslikava iz $G \times H$ v G, definirana z $(g, h) \to g$, homomorfizem. Določi jedro homomorfizma. Ta preslikava se imenuje projekcija grupe $G \times H$ na grupo G.

(ii) Pokaži, da je preslikava $\eta : G \to G \times H$, definirana z $\eta(g) = (g, e_H)$, homomorfizem (e_h je identiteta grupe H). Določi jedro tega homomorfizma.

Izrek (lastnosti elementov glede na homomorfizem)

Naj bo ϕ homomorfizem iz grupe G v grupo $\overline{G}$, in naj bo g element grupe G. Potem

1. ϕ preslika identiteto grupe G v identiteto grupe $\overline{G}$.
2. $\phi(g^n) = (\phi(g))^n$ za vsak $n \in \mathbb{Z}$.
3. Če je $|g|$ končen, potem $|\phi(g)|$ deli $|g|$.
4. kerϕ je podgrupa grupe G.
5. $\phi(a) = \phi(b)$ če in samo če $a$ker$\phi = b$kerϕ.
6. Če je $\phi(g) = g'$, potem je $\phi^{-1}(g') = \{x \in G \,|\, \phi(x) = g'\} = gker\phi$.

5. Naj bosta G in $\overline{G}$ dve končni grupi in naj bo $\phi : G \to \overline{G}$ surjektivni homomorfizem. Pokaži, da če ima $\overline{G}$ element reda n, potem ima tudi G element reda n.

Izrek (lastnosti podgrupe glede na homomorfizem)

Naj bo ϕ homomorfizem iz grupe G v grupo $\overline{G}$, in naj bo H podgrupa grupe G. Potem

1. $\phi(H) = \{\phi(h) \,|\, h \in H\}$ je podgrupa grupe $\overline{G}$.
2. Če je H ciklična, potem je $\phi(H)$ ciklična.
3. Če je H, abelska potem je $\phi(H)$ abelska.
4. Če je H edinka v G, potem je $\phi(H)$ edinka v $\phi(G)$.
5. Če je $|\mathrm{ker}\phi| = n$, potem je ϕ n-na-1 preslikava iz G na $\phi(G)$ (ϕ je n-na-1 surjekcija).
6. Če je $|H| = n$, potem $|\phi(H)|$ deli n.
7. Če je $\overline{K}$ podgrupa grupe $\overline{G}$, potem je $\phi^{-1}(\overline{K}) = \{k \in G \,|\, \phi(k) \in \overline{K}\}$ podgrupa grupe G.
8. Če je $\overline{K}$ edinka grupe $\overline{G}$, potem je $\phi^{-1}(\overline{K}) = \{k \in G \,|\, \phi(k) \in \overline{K}\}$ edinka grupe G.
9. Če je ϕ surjekcija in ker$\phi = \{e\}$, potem je ϕ izomorfizem iz G v $\overline{G}$

6. Poišči netrivialni homomorfizem ϕ iz G v $\overline{G}$ (to pomeni, poišči tak homomorfizem, da je $\phi(G) \neq \{e\}$). Za vsak primer pokaži, da je ϕ homomorfizem ter določi njegovo jedro.

(i) $G = \mathbb{Z}_{35}$, $\overline{G} = \mathbb{Z}_5$.

(ii) $G = \mathbb{Z}_5$, $\overline{G} = \mathbb{Z}_{35}$.

(iii) $G = (\mathbb{R}, +)$, $\overline{G} = \mathrm{GL}_2(\mathbb{R})$.

7. (i) Pokaži, da ne obstaja netrivialni homomorfizem iz A_5 v $\mathbb{Z}_7 \times \mathbb{Z}_7$.

(ii) Pokaži, da ne obstaja surjektivni homomorfizem iz $\mathbb{Z}_8 \times \mathbb{Z}_2$ v $\mathbb{Z}_4 \times \mathbb{Z}_4$.

8. Ali je mogoče, da obstaja homomorfizem iz $\mathbb{Z}_4 \times \mathbb{Z}_4$ na $\mathbb{Z}_8$ (tj. da obstaja surjektivni homomorfizem iz $\mathbb{Z}_4 \times \mathbb{Z}_4$ v $\mathbb{Z}_8$)? Ali je mogoče, da obstaja homomorfizem iz $\mathbb{Z}_{16}$ na $\mathbb{Z}_2 \times \mathbb{Z}_2$? Odgovor utemelji!

9. Predpostavimo, da obstaja homomorfizem iz končne grupe G na $\mathbb{Z}_{10}$ (tj. da obstaja surjektivni homomorfizem iz končne grupe G v grupo $\mathbb{Z}_{10}$). Pokaži, da ima potem G edinki indeksa 2 in 5.

Izrek (jedro homomorfizma je edinka)
Naj bo ϕ homomorfizem iz grupe G v grupo $\overline{G}$. Potem je $\ker\phi$ edinka grupe G.

Izrek (prvi izrek o izomorfizmu (Jordan, 1870))
Naj bo ϕ homomorfizem grupa iz grupe G v grupo $\overline{G}$. Potem je preslikava iz grupe $G/\ker\phi$ v grupo $\phi(G)$, podana z $g\ker\phi \to \phi(g)$, izomorfizam. Torej $G/\ker\phi \cong \phi(G)$.

Posledica
Naj bo ϕ homomorfizem iz končne grupe G v grupo $\overline{G}$. Potem $|\phi(G)|$ deli $|G|$ in $|\overline{G}|$.

10. Uporabi prvi izrek o izomorfizmu, in pokaži da je

(i) $\mathbb{Z} \times \mathbb{Z}/\langle(1,1)\rangle \cong \mathbb{Z}$.

(ii) $\mathbb{Z} \times \mathbb{Z}/\langle(1,5)\rangle \cong \mathbb{Z}$.

11. Razloži, zakaj $\mathbb{Z} \times \mathbb{Z}/\langle(2,4)$ ni izomorfna z grupo $\mathbb{Z}$.

12. Naj bodo G, H in K dane grupe. Naj bo ϕ homomorfizem iz G v H in naj bo σ homomorfizem iz H v K. Pokaži, da je $\sigma\phi$ homomorfizem iz G v K. Na kakšen način sta $\ker\phi$ in $\ker\sigma\phi$ povezana? Če sta ϕ in σ surjektivna homomorfizma in če je G končno, opiši $[\ker\sigma\phi : \ker\phi]$ preko $|H|$ in $|K|$.

13. Prevzemimo, da k deli n. Pokaži, da je $\mathbb{Z}_n/\langle k\rangle \cong \mathbb{Z}_k$.

14. Naj bo N edinka končne grupe G.

(i) Pokaži, da red elementa $gN \in G/N$ deli red elementa g.

(ii) Pokaži, da je vsaka podgrupa grupe G/N oblike H/N, kje je H podgrupa grupe G.

15. Pokaži, da je vsaka grupa reda 77 ciklična.

16. Določi vse homomorfizme iz $\mathbb{Z}$ na S_3 (tj. vse surjektivne homomorfizme). Določi vse homomorfizme iz $\mathbb{Z}$ v S_3.

17. Naj bo p praštevilo. Določi število homomorfizmov iz $\mathbb{Z}_p \times \mathbb{Z}_p$ v $\mathbb{Z}_p$.

18. Za vsak par pozitivnih celih števil m in n, lahko definiramo homomorfizem iz $\mathbb{Z}$ v $\mathbb{Z}_m \times \mathbb{Z}_n$ s

$$x \longrightarrow (x \bmod m, x \bmod n).$$

Določi jedro tega homomorfizma v primeru, ko je $(m,n) = (3,4)$. Določi jedro tega homomorfizma v primeru, ko je $(m,n) = (6,4)$. Rezultat posploši.

19. Naj bosta m, n dve tuji celi števili. Definirajmo preslikavo $\phi : \mathbb{Z} \to \mathbb{Z}_m \times \mathbb{Z}_n$ s $\phi(a) = (a \bmod m, a \bmod n)$.

(i) Pokaži, da je ϕ homomorfizem grup.

(ii) Uporabi prvi izrek o izomorfizmu in pokaži, da je $\mathbb{Z}_{mn} \cong \mathbb{Z}_m \times \mathbb{Z}_n$.

(iii) Uporabi (ii) in pokaži nasljednje: Dokaži, da obstaja celo število x tako da je $x \equiv 3 \bmod 5$ in $x \equiv 12 \bmod 17$.

Serge Lang

> Lang's Algebra changed the way graduate algebra is taught... It has affected all subsequent graduate-level algebra books.
>
> *Citation for the Steele Prize*

Serge Lang was a prolific mathematician, inspiring teacher, and political activist. He was born near Paris on May 19, 1927. His family moved to Los Angeles when he was a teenager. Lang received a B.A. in physics from Caltech in 1946 and a Ph.D. in mathematics from Princeton in 1951 under Emil Artin. His first permanent position was at Columbia University in 1955, but in 1971 Lang resigned his position at Columbia as a protest against Columbia's handling of Vietnam antiwar protesters. He joined Yale University in 1972 and remained there until his retirement.

Lang made significant contributions to number theory, algebraic geometry, differential geometry, and analysis. He wrote more than 120 research articles and 60 books. His most famous and influential book was his graduate-level Algebra. Lang was a prize-winning teacher known for his extraordinary devotion to students. Lang often got into heated discussions about mathematics, the arts, and politics. In one incident, he threatened to hit a fellow mathematician with a bronze bust for not conceding it was self-evident that the Beatles were greater musicians than Beethoven.

Among Lang's honors were the Steele Prize for Mathematical Exposition from the American Mathematical Society, the Cole Prize in Algebra, and election to the National Academy of Sciences. Lang died on September 25, 2005, at the age of 78.

On the Google Drive, among else, please find solutions for the following problems:

1. Suppose that ϕ is a homomorphism from a finite group G onto $\overline{G}$ and that $\overline{G}$ has an element of order 8. Prove that G has an element of order 8. Generalize. **2.** Suppose that $f : G \to Q$ is a homomorphism and that $H \lhd G$ which satisfies $H \subseteq \ker(f)$. Prove that $\widetilde{f} : G/H \to Q$ defined by $\widetilde{f}(aH) = f(a)$ is well-defined and a homomorphism. (Note: you can describe this phenomenon in a couple of ways. Some people say $\widetilde{f}$ is the homomorphism on G/H induced by f; others say that f descends to the homomorphism $\widetilde{f}$ on G/H.) **3.** Suppose that G is a group so that $[G : Z(G)] \leq 3$. Prove that G is abelian. **4.** Prove that a non-trivial, finite abelian group G is simple if and only if $G \cong \mathbb{Z}_p$ for some prime p. (Recall that a group G is called simple if the only normal subgroups it contains are $\{G\}$ and G.) **5.** Prove the second isomorphism theorem: If $K \leq G$ and $N \lhd G$, then $K/(K \cap N) \cong KN/N$. (Note: The set KN is defined as $KN = \{kn : k \in K \text{ and } n \in N\}$. For this statement to make sense you should check that $N \lhd KN$.) **6.** Prove the third isomorphism theorem: if H, K are normal subgroups of G with $H \leq K \leq G$, then $\frac{G/H}{K/H} \cong G/K$.

Computer Tutorial 11.[29][30]

subspaces of vector spaces

<table>
<tr><th>Input</th><th>Meaning</th></tr>
<tr><td>R := RealField();
V := VectorSpace(R,5);
a1 := V![1,1,0,1,1];
a2 := V![0,1,0,1,-1];
a3 := V![3,1,0,1,5];
W := sub< V | a1, a2, a3 >;
Dimension(W);</td><td>Define the real field R := RealField(); .
Create a vector space V of dimension 5 over R.
Create the subspace of $\mathbb{R}^5$ spanned by the vectors $(1,1,0,1,1)^\top$, $(0,1,0,1,-1)^\top$ and $(3,1,0,1,5)^\top$.

Find the dimension of subspace W. Are the vectors a1, a2 and a3 linearly independent?</td></tr>
<tr><td>b1 := V![1,1,0,1,1];
b2 := V![0,1,0,1,0];
b3 := V![-1,2,-3,4,-5];
v := V![1,5,-4,3,6];

M := KMatrixSpace(R,3,5);
B := M![b1,b2,b3];
B, v;
A := Transpose(B);

x := Solution(B*A, v*A);
x;

x * B;</td><td>Consider the following vectors in $\mathbb{R}^5$:
$b_1 = (1,1,0,1,1)^\top$, $b_2 = (0,1,0,1,0)^\top$,
$b_3 = (-1,2,-3,4,-5)^\top$, $v = (1,5,-4,3,6)^\top$.
We wish to calculate the projection of v onto the subspace W of $\mathbb{R}^5$ with basis $\{b_1, b_2, b_3\}$. Since MAGMA uses row vectors rather than column vectors, we will transpose everything.
Create the vector space of 3×5 matrices.
Create the matrix B with rows $b_1^\top$, $b_2^\top$ and $b_3^\top$. (This is the transpose of the matrix A whose columns are b_1, b_2 and b_3).
Print B and v to check that you have entered everything correctly, and then define A:=Transpose(B);.
According to the theory, the projection of v onto W is the vector Ax, where $A^\top Ax = A^\top v$. Taking transposes this equation becomes $x^\top A^\top A = v^\top A$, since transposing reverses multiplication. We can obtain the vector $x^\top$ via the command x:=Solution(B*A,v*A);. (Explanation: if M is a matrix and b a (row) vector then Solution(M,b) is a vector x that is a solution of the matrix equation $xM = b$.)
The projection is $p = Ax$. Transposing (remembering that $B = A^\top$ and that the vector x that MAGMA has found is actually $x^\top$), we see that x*B; will print out the row vector that is the transpose of p.</td></tr>
<tr><td>R := RealField(10);
V := VectorSpace(R,6);
M := KMatrixSpace(R,2,6);
B := M![1,1,1,1,1,1,104,105,
106,107,108,109] where
M is KMatrixSpace(R,2,6);
y := V![1.8,1.5,-0.05,-2.7,-2.5];
x := Solution(B*A, y*A) where A is
Transpose(B);
x;
U := VectorSpace(R,2);
a := U![1,110];
InnerProduct(x,a);
10^(-3.98866153);</td><td>Find the least squares line of best fit for the following data. (atomic no.; $\log_{10}$ of half-life (seconds)): (104; 1.8), (105; 1.5), (106; -0.05), (107; -0.97), (108; -2.7), (109; -2.5). There are six data points $(x_1; y_1)$, ..., $(x_6; y_6)$. By the theory the line of best fit is $y = a + bx$, where $a(1,1,1,1,1,1) + b(x_1, x_2, x_3, x_4, x_5, x_6)$ is as close as possible to $(y_1, y_2, y_3, y_4, y_5, y_6)$. Now make a prediction for the half-life of element number 110.
Recall that the value of x gives the coefficients α and β in the equation $y = \alpha + \beta x$ of the line of best fit. So in this case the line is $y = 106.074 - x$. Evaluating the right hand-side at $x = 110$ amounts to calculating the dot product of $(106.074, -1)$ and $(1, 110)$, and this explains the three MAGMA commands above following the calculation of x. Note that $-3.988...$ is the log of the half-life; to get the half-life itself we must raise 10 to this power.
If the prediction is correct, the half-line is about one ten-thousandth of a second.</td></tr>
</table>

[29]To write MAGMA code please open: http://magma.maths.usyd.edu.au/calc/

[30]See also: http://www.maths.usyd.edu.au/u/bobh/UoS/MATH2008/ctut11.pdf